

A Note on Discrete Groups*

S. O. Juriaans, S.C. Lima Neto, A. De A. E Silva

Abstract

We prove that a Kleinian groups has a DF domain if and only if it has a DC domain . The Fuchsian case has recently been considered, it was shown that, in this case, there are no cocompact examples and cocompact Kleinian examples were given. Here we prove that, in the Kleinian case, there are no cocompact torsion free examples and we describe the symmetries of a fundamental domain of such a group.

1 Introduction

In [5] it is proved that in hyperbolic 2 and 3-space the isometric spheres, in the ball models, are also the Poincaré bisectors. This was used to get explicit formulas for the Poincaré bisectors in hyperbolic 2 and 3-space. Using these formulas, generators were found for discrete groups of quaternions division algebras and Poincaré fundamental polygons were constructed for the Bianchi groups and the Figure Eight Knot group. Note that in [6, 7, 10] similar questions are addressed.

Two interesting problems are that of deciding when a Ford fundamental domain coincides with a Poincaré fundamental domain (called a Dirichlet-Ford domain or DF domains) and when a Poincaré fundamental domain has more than one center (called a Double Dirichlet domain or DC domain). These problems were raised in [8] and solved, in the same paper, for Fuchsian groups. In particular it is proved that there are no cocompact examples in this case. In [5] an independent proof was given and an algebraic criterium was established which the set of side-pairing transformations must satisfy. Actually it turns out that, in the Fuchsian case, these two problems have identical solutions ([8]) and the question remained to see what happens in the Kleinian case.

Our main result in this paper is to solve above mentioned problems for Kleinian groups. In particular, we show that also in this case, they are identical. Cocompact examples are constructed in [8] and here we show that no cocompact torsion free examples exist. The difference with the Fuchsian case lies in the possible number of linear orthogonal maps A which arise as one writes a hyperbolic isometry $\gamma = A\sigma$, where σ is the reflection in the isometric sphere. In the Fuchsian case only one such reflections shows up, namely the reflection in the imaginary axis. In the Kleinian case, we first give a rather good description of A and use this to show that in a DF domain all of this linear maps, coming from the sidepairing transformations, have a common eigenvector. If the group is torsion free then the direction of this eigenvector determines an ideal vertex. Together with the results of [5] this gives an algebraic characterization, in term of a set of sidepairing transformations, of the Kleinian groups having a DF domain. A part from from solving the above mentioned

*Mathematics subject Classification Primary [30F40]; Secondary [20H10]. Keywords and phrases: Hyperbolic space, Isometric sphere, Bisector, Canonical region.
Research partially supported by CNPq, UFPB and UNIVASF.

problems for Kleinian groups, we study the symmetry of their fundamental domain and derive some consequences of the fact that the isometric spheres, in the ball models of hyperbolic space, are the Poincaré bisectors in any dimension.

The layout of the paper is as follows. In Section 2 we recall fundamentals and also some results of [5] that we will need in the sequel. In Section 3 we settle the DF and DC problems for Kleinian groups. In Section 4, we study the symmetries of a Kleinian group having a DF domain and make some considerations on Kleinian groups and the bisectors of their elements. Most of the notation used is standard or follows that introduced in [5].

2 Poincaré Bisectors

In this section we recall basic facts on hyperbolic spaces, fix notation and generalize a result of [5]. Standard references are [1, 2, 3, 4, 9]. By \mathbb{H}^n (respectively \mathbb{B}^n) we denote the upper half space (plane) (respectively the ball) model of hyperbolic n -space.

The hyperbolic distance ρ in \mathbb{H}^3 is determined by $\cosh \rho(P, P') = \delta(P, P') = 1 + \frac{d(P, P')^2}{2rr'}$, where d is the Euclidean distance and $P = z + rj$, $P' = z' + r'j$ are two elements of \mathbb{H}^3 .

Let Γ be a discrete subgroup of $\text{Iso}^+(\mathbb{B}^3)$. The Poincaré method can be used to give a presentation of Γ (see for example [9]). Let Γ_0 be the stabilizer in Γ of $0 \in \mathbb{B}^3$ and let \mathcal{F}_0 a fundamental domain for Γ_0 . For $\gamma \in \text{Iso}^+(\mathbb{B}^3)$, let $D_\gamma(0) = \{u \in \mathbb{B}^3 \mid \rho(0, u) \leq \rho(u, \gamma(0))\}$ and $\Sigma_\gamma = \{u \in \mathbb{B}^3 \mid \rho(0, u) = \rho(u, \gamma(0))\}$ (see [5]). Then $\mathcal{F} = \mathcal{F}_0 \cap (\bigcap_{\gamma \in \Gamma \setminus \Gamma_0} D_\gamma(0))$ is a Poincaré fundamental domain of Γ with center 0. If $\Gamma_0 = 1$ then 0 is in the interior of the fundamental domain.

If S_1 and S_2 are two intersecting spheres in the extended hyperbolic space, then (S_1, S_2) denotes the cosine of the angle at which they intersect, the dihedral angle. This notation is taken from [1]. Elements x and y of hyperbolic space are inverse points with respect to S_1 if $y = \sigma(x)$, where σ is the reflection in S_1 . In case $S_1 = \partial \mathbb{B}^3 = S^2$, the boundary of \mathbb{B}^3 , then the inverse point of x with respect to S_1 is denoted by x^* .

Let $\gamma \in \text{PSL}(2, \mathbb{C})$, $z_0 \in \mathbb{B}^3$ and $\Psi : \text{PSL}(2, \mathbb{C}) \rightarrow \text{Iso}^+(\mathbb{B}^3)$ an isomorphism. One can identify $\text{Iso}^+(\mathbb{B}^3)$ with a subgroup of two by two matrices over the quaternions over the reals (see [3]). Define $\Sigma_{\Psi(\gamma)}(z_0) := \{u \in \mathbb{B}^3 \mid \rho(z_0, u) = \rho(u, \Psi(\gamma)^{-1}(z_0))\}$. Clearly $\Sigma_{\Psi(\gamma)}(0) = \Sigma_{\Psi(\gamma)}$ and $\Psi(\gamma_1)(\Sigma_{\Psi(\gamma_1^{-1}\gamma\gamma_1)}(z_0)) = \Sigma_{\Psi(\gamma)}(\gamma_1(0))$. If Γ is a Kleinian group then define $D_\Gamma(z_0)$ as the intersection of \mathbb{B}^3 and the closure of $\bigcap_{\gamma \in \Gamma} \text{Exterior}(\Sigma_{\Psi(\gamma)}(z_0))$. We have that $D_\Gamma = D_\Gamma(0)$ and

$$D_\Gamma(\gamma_1(0)) = \gamma_1(D_{\gamma_1^{-1}\Gamma\gamma_1}).$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathbb{C})$, write $a = a(\gamma)$, $b = b(\gamma)$, $c = c(\gamma)$ and $d = d(\gamma)$ when it is necessary to stress the dependence of the entries on the matrix γ . It is known that $\Psi(\gamma) = A_{\Psi(\gamma)}\sigma_{\Psi(\gamma)}$, where $A_{\Psi(\gamma)}$ is a linear orthogonal map and $\sigma_{\Psi(\gamma)}$ is the reflection in the isometric sphere of $\Psi(\gamma)$.

Given $z_0 \in \mathbb{B}^3$, let $P_{z_0} = z_0^*$ be the inverse point of z_0 with respect to $S_1^2(0)$. Choose $R_{z_0} > 0$ such that $1 + R_{z_0}^2 = \|P_{z_0}\|^2$, let $\Sigma_{z_0} = S_{R_{z_0}}(P_{z_0})$ and let σ_{z_0} be the reflection in Σ_{z_0} . Let $W_{z_0} = \text{span}_{\mathbb{R}}[j, z_0]$ be the plane spanned by j and z_0 , let A_{z_0} be the reflection in W_{z_0} and let $\gamma_{z_0} = A_{z_0} \circ \sigma_{z_0}$. It is easily seen that γ_{z_0} is an orientation preserving isometry of \mathbb{B}^3 , $\rho(\gamma_{z_0}) = 2$, $\gamma_{z_0}(z_0) = z_0$ and $\Sigma_{\gamma_{z_0}} = \Sigma_{z_0}$. Hence $D_\Gamma = \gamma_{z_0}(D_{\gamma_{z_0}\Gamma\gamma_{z_0}})$ if and only if $D_\Gamma = D_\Gamma(z_0)$ if and only if $D_{\gamma_{z_0}\Gamma\gamma_{z_0}} = \gamma_{z_0}(D_\Gamma)$.

We next recall some results proved in [5] which will be needed in the sequel. Some are just partial statements of the complete results. The first result we state identifies the Poincaré bisectors in the ball model of hyperbolic 3-space.

Theorem 2.1 *Let $\gamma \in \text{SL}(2, \mathbb{C})$ with $\gamma \notin \text{SU}(2, \mathbb{C})$.*

Then $\Sigma_{\Psi(\gamma)} = \{u \in \mathbb{B}^3 \mid \rho(0, u) = \rho(u, \Psi(\gamma^{-1})(0))\}$, the bisector of the geodesic segment linking 0 and $\Psi(\gamma^{-1})(0)$, is the isometric sphere of $\Psi(\gamma)$. Moreover $1 + \frac{1}{|C|^2} = |P_{\Psi(\gamma)}|^2$, $D_\gamma(0) = \mathbb{B} \cap \text{Exterior}(\Sigma_{\Psi(\gamma)})$ and $P_{\Psi(\gamma)}^ = \Psi(\gamma^{-1})(0)$.*

Using this result and the theory of hyperbolic spaces one gets explicit formulas for the Poincaré bisectors in the upper half space model. In fact, we have the following results from [5].

Proposition 2.2 *Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})$ and $\Sigma_\gamma = \eta_0^{-1}(\Sigma_{\Psi(\gamma)})$, where $\eta_0 : \mathbb{H}^3 \rightarrow \mathbb{B}^3$ is an isometry between the models (see [3]).*

1. Σ_γ is an Euclidean sphere if and only if $|a|^2 + |c|^2 \neq 1$. In this case, its center and its radius are respectively given by $P_\gamma = \frac{-(\bar{a}b + \bar{c}d)}{|a|^2 + |c|^2 - 1}$, $R_\gamma^2 = \frac{1 + \|P_\gamma\|^2}{|a|^2 + |c|^2}$.
2. Σ_γ is a plane if and only if $|a|^2 + |c|^2 = 1$. In this case $\text{Re}(\bar{v}z) + \frac{|v|^2}{2} = 0$, $z \in \mathbb{C}$ is a defining equation of Σ_γ , where $v = \bar{a}b + \bar{c}d$.
3. $|\bar{a}b + \bar{c}d|^2 = (|a|^2 + |c|^2)(|b|^2 + |d|^2) - 1$
4. Suppose $c \neq 0$. Then $|\hat{P}_\gamma - P_\gamma| = \frac{|d - \bar{a}|}{|c|(|a|^2 + |c|^2 - 1)}$. Moreover $\text{ISO}_\gamma = \Sigma_\gamma$ if and only if $d = \bar{a}$. In this case we also have that $c = \lambda \bar{b}$, with $\lambda \in \mathbb{R}$. If $c = 0$ and $\infty \in \Sigma_\gamma$ then the same conclusion holds.
5. Suppose that $\text{ISO}_\gamma = \Sigma_\gamma$ or that $c = 0$ and $\infty \in \Sigma_\gamma$. Then $\text{tr}(\gamma) \in \mathbb{R}$

Proposition 2.3 *Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})$ and $\Psi(\gamma) = \begin{pmatrix} A & C' \\ C & A' \end{pmatrix}$. Then the following properties hold.*

1. $\text{ISO}_{\Psi(\gamma)} = \Sigma_{\Psi(\gamma)}$
2. $|A|^2 = \frac{2 + \|\gamma\|^2}{4}$, $|C|^2 = \frac{\|\gamma\|^2 - 2}{4}$ and $|A|^2 - |C|^2 = 1$
3. $P_{\Psi(\gamma)} = \frac{1}{-2 + \|\gamma\|^2} \cdot [-2(\bar{a}b + \bar{c}d) + [(|b|^2 + |d|^2) - (|a|^2 + |c|^2)]j]$
4. $\Psi(\gamma^{-1})(0) = P_{\Psi(\gamma)}^* = \frac{1}{2 + \|\gamma\|^2} \cdot [-2(\bar{a}b + \bar{c}d) + [(|b|^2 + |d|^2) - (|a|^2 + |c|^2)]j]$ (notation of inverse point w.r.t. S^2).
5. $\|P_{\Psi(\gamma)}\|^2 = \frac{2 + \|\gamma\|^2}{-2 + \|\gamma\|^2}$
6. $R_{\Psi(\gamma)}^2 = \frac{4}{-2 + \|\gamma\|^2}$
7. $\Sigma_{\Psi(\gamma)} = \Sigma_{\Psi(\gamma_1)}$ if and only if $\gamma_1 = \gamma_0 \gamma$, $\gamma_0 \in \text{SU}(2, \mathbb{C})$.

We will need to calculate the dihedral angle between two bisectors. For this we state the following result also obtained in [5].

Lemma 2.4 *Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})$, $\pi_0 = \partial \mathbb{H}^n$, $n = 2, 3$, and θ the angle between $\Sigma_{\Psi(\gamma)}$ and $\Sigma = \Sigma_{\Psi(\gamma_1)}$ with $\Sigma \cap \Sigma_{\Psi(\gamma)} \neq \emptyset$. Then $\cos(\theta) = \frac{|1 - \langle P_{\Psi(\gamma)} | P_{\Psi(\gamma_1)} \rangle|}{R_{\Psi(\gamma_1)} \cdot R_{\Psi(\gamma)}}$.*

Our next result proves that [5, Theorem 3.1] holds in all dimensions. With this result at hand we can now work in both models and carry over information in a simple way.

Theorem 2.5 *Let γ be an orientation preserving isometry of \mathbb{B}^n and let Σ_γ be its isometric sphere. Then Σ_γ is the bisector of geodesic linking the origin 0 and $\gamma^{-1}(0)$.*

Proof. Write $\gamma = A\sigma$, where A is an orthogonal map and σ the reflection in Σ_γ . Then $\gamma^{-1}(0) = \sigma A^{-1}(0) = \sigma(0)$ and hence 0 and $\sigma(0)$ are inverse point with respect to Σ_γ . From this it follows that Σ_γ is the bisector of the geodesic linking 0 and $\sigma(0)$. ■

When looking for relations it is necessary to know the position of the bisectors relative to one another. In this direction it is easy to see that if $\gamma, \gamma_1 \in \text{PSL}(2, \mathbb{C})$ are non-unitary and that $\gamma\gamma_1$ is also non-unitary then $\gamma_1^{-1}(\Sigma_\gamma) \cap \Sigma_{\gamma\gamma_1} = \Sigma_\gamma \cap \Sigma_{\gamma_1}$. From the latter it follows that $\gamma_1(\gamma_1(\Sigma_\gamma) \cap \Sigma_{\gamma\gamma_1}) = \Sigma_{\gamma_1^{-1}} \cap \Sigma_\gamma$. The Poincaré Theory tells us how to find relations.

3 DF and DC Domains

In this section we consider Kleinian groups which have a double Dirichlet domain or a Dirichlet-Ford domain. In particular, we settle a question on these groups raised in [8].

Let Γ be a Kleinian group and $\gamma \in \Gamma$. Write $\gamma = A_\gamma \sigma_\gamma$ where σ_γ is the reflection in Σ_γ . Note that we have that $A_\gamma(\Sigma_\gamma) = \Sigma_{\gamma^{-1}}$. Since j and $\gamma^{-1}(j)$ are inverse points with respect to Σ_γ we have that $A_\gamma(j) = j$.

Lemma 3.1 *Let $\gamma \in \Gamma$. Then the following hold.*

1. $A_\gamma(j) = j$
2. $A_\gamma(P_\gamma) = \hat{P}_\gamma$
3. $A_\gamma(\infty) = \frac{b(\gamma) + \overline{c(\gamma)}}{d(\gamma) - \overline{a(\gamma)}}$, if $d(\gamma) \neq \overline{a(\gamma)}$
4. $A_\gamma(0) = \frac{b(|a|^2 + |c|^2 - 1) - \overline{c}(|b|^2 + |d|^2 - 1)}{d(|a|^2 + |c|^2 - 1) + \overline{a}(|b|^2 + |d|^2 - 1)}$
5. If $d(\gamma) = \overline{a(\gamma)}$ and $\gamma \notin \Gamma_j$ then $A_\gamma(\infty) = \infty$ and $A_\gamma(0) = 0$.

Proof. The first item, as seen above, is obvious. We have that $A_\gamma(P_\gamma) = \gamma \circ \sigma_\gamma(P_\gamma) = \gamma(\infty) = \hat{P}_\gamma$. To prove the third item notice that $A_\gamma(\infty) = \gamma(P_\gamma)$. Using the expression of P_γ and that $\det(\gamma) = 1$, the third item follows. To prove the fourth item, note that $\sigma_\gamma(P) = P_\gamma + \frac{R_\gamma^2}{\|P - P_\gamma\|}(P - P_\gamma)$,

$\sigma_\gamma(0) = \frac{\|P_\gamma\|^2 - R_\gamma^2}{\|P_\gamma\|^2} \cdot P_\gamma$ and hence $A_\gamma(0) = \gamma(\frac{\|P_\gamma\|^2 - R_\gamma^2}{\|P_\gamma\|^2} \cdot P_\gamma)$. From this the fourth item follows easily.

We now prove the last item. We have to consider all possible situations but apart from this the proof is straightforward.

We first suppose that $d = d(\gamma) = 0$. In this case $c = c(\gamma) \neq 0$. If $\gamma \notin \Gamma_j$ then we have that $P_\gamma = \hat{P}_\gamma = 0$ and $\sigma_\gamma(\infty) = 0$. From this we have that $A_\gamma(\infty) = \gamma(0) = \infty$ and $A_\gamma(0) = \gamma(\infty) = 0$.

If $d \neq 0$ and $c \neq 0$ then $P_\gamma = \hat{P}_\gamma$ and $\sigma_\gamma(P) = P_\gamma + \frac{R_\gamma^2}{\|P - P_\gamma\|}(P - P_\gamma)$. From this we have that $A_\gamma(0) = \gamma(\frac{1 - |d|^2}{cd}) = 0$ and $A_\gamma(\infty) = \gamma(\hat{P}_\gamma) = \infty$.

If $d \neq 0$ and $c = 0$ then ISO_γ does not exist, $|a| = 1$ and hence Σ_γ is a vertical plane. Using the defining equation of Σ_γ we have that $\sigma_\gamma(0) = -(b\bar{a} + d\bar{c}) = -bd$. Hence $A_\gamma(0) = \gamma(-bd) = 0$ and $A_\gamma(\infty) = \gamma(\infty) = \infty$. ■

Observe that $A_\gamma = \eta_0^{-1} \circ A_{\Psi(\gamma)} \circ \eta_0$ and $\sigma_\gamma = \eta_0^{-1} \circ \sigma_{\Psi(\gamma)} \circ \eta_0$. Hence, working in \mathbb{B}^3 , we see that $A_{\gamma^{-1}} = A_\gamma^{-1}$ and $\sigma_{\gamma^{-1}} = A_\gamma \circ \sigma_\gamma \circ A_\gamma^{-1}$.

Theorem 3.2 *Let γ be a Kleinian group. The following statements are equivalent.*

1. *There exist $z_0 \neq z_1$ such that $D_\Gamma(z_0) = D_\Gamma(z_1)$.*
2. *Γ has a Dirichlet-Ford domain.*

Moreover, if Γ is torsion free then it is not cocompact.

Proof. Suppose first that Γ has a Double Dirichlet domain and let Φ be the set of side-pairing transformations of Γ . We will work first in the ball model but, for simplicity, keep the notion of the upper half plane model. We may suppose that $z_1 = 0$ and Φ is taken with respect to D_Γ . Our hypothesis is that $D_\Gamma = D_\Gamma(z_0)$. Hence, given $\gamma \in \Phi$ there exists $\gamma_1 \in \Gamma$ such that $\Sigma_\gamma = \Sigma_{\gamma_1}(z_0)$. In particular z_0 and $\gamma_1^{-1}(z_0)$ are inverse points with respect to Σ_γ and thus $\gamma_1^{-1}(z_0) = \sigma_\gamma(z_0)$. From this we obtain that $\gamma(\gamma_1^{-1}(z_0)) = \gamma\sigma(z_0) = A_\gamma(z_0)$. Consequently, $\|\gamma(\gamma_1^{-1}(z_0))\| = \|A_\gamma(z_0)\| = \|z_0\|$ and hence, by [9, Theorem IV.5.1], we have that $0 \in \Sigma_{\gamma_1\gamma^{-1}}(z_0)$. But 0 belongs to the interior of $D_\Gamma(z_0) = D_\Gamma$; hence we have a contradiction unless $\gamma_1\gamma \in \Gamma_{z_0}$, i.e., $\Sigma_{\gamma_1\gamma}(z_0)$ does not exist. So we proved that $A_\gamma(z_0) = z_0$ for every $\gamma \in \Phi$. If $\lambda > 0$, with $\lambda z_0 \in D_\Gamma$ and $\gamma \in \Phi$, we have that $\gamma^{-1}(\lambda z_0) = \sigma_\gamma(A_\gamma^{-1}(\lambda z_0)) = \sigma_\gamma(\lambda z_0)$, i.e., λz_0 and $\gamma^{-1}(\lambda z_0)$ are inverse points with respect to Σ_γ . Hence we proved that $\Sigma_\gamma = \Sigma_\gamma(\lambda z_0)$. To complete this part of the proof, we now work in \mathbb{H}^3 and suppose that $z_0 = j$. So we have that $A_\gamma(\lambda j) = \lambda j$, for all $\lambda > 0$. By continuity, it follows that $A_\gamma(\infty) = \infty$. By Lemma 3.1, we have that $\Sigma_\gamma = \text{ISO}_\gamma$.

Now suppose that Γ is torsion free. Let $V_\gamma = \mathbb{B}^3 \cap \Sigma_\gamma \cap \mathbb{R}z_0$ and suppose that $V_\gamma = \{z_\gamma\}$. Then clearly $A_\gamma(z_\gamma) = z_\gamma$ and hence $\gamma(z_\gamma) = z_\gamma$. It follows that $o(\gamma) < \infty$ and Σ_γ does not exist, a contradiction. So we have that $V_\gamma = \emptyset$ for all $\gamma \in \Phi$. From this it follows that $S^2 \cap \mathbb{R}z_0$ is not covered by any Σ_γ and hence Γ is not cocompact.

We now prove the converse. Suppose that Γ has a Dirichlet-Ford domain and let Φ be the set of side-pairing transformations of Γ . For every $\gamma \in \Phi$ we have that $d(\gamma) = \overline{a(\gamma)}$ ([5] or Proposition 2.2) and hence, by Lemma 3.1, we have that $A_\gamma(0) = 0$. In particular, A_γ is an Euclidean linear isometry and hence $A_\gamma(\lambda j) = \lambda j$, for all $\lambda > 0$. From this we have that for every $\lambda > 0$, with λj an interior point of the domain, $\gamma^{-1}(\lambda j) = \sigma_\gamma(\lambda j)$, i.e., $\Sigma_\gamma(\lambda j) = \Sigma_\gamma(j) = \Sigma_\gamma$. ■

Together with the results of Section IV of [5], this theorem gives a complete characterization of DF and DC domains and gives an algebraic criterium to decide whether a domain is a DF domain (and hence a DC domain).

4 Bisector of Kleinian Groups

In this section we will frequently switch between the ball and upper half space models of hyperbolic 2 and 3-space and Γ will stand for a discrete group of orientation preserving isometries. We keep notation as simple as possible to avoid confusion. Note that the results presented here for Kleinian groups are similar to those in [1] for Fuchsian groups. A theory of pencils can also be developed in this case.

We first consider hyperbolic 3-space. We work in the ball model but, for simplicity, use the notation of the upper half plane model. Let $\gamma \in \Gamma$ and write $\gamma = A_\gamma \sigma_\gamma$. It follows that $\det(A_\gamma) = -1$ and hence there exists $p_\gamma \in \mathbb{B}^3$, such that $A_\gamma(p_\gamma) = -p_\gamma$. We have that $A_\gamma(P_\gamma) = \gamma(\infty) = P_{\gamma^{-1}}$. Since A_γ is a linear orthogonal map we obtain that $\langle p_\gamma | P_\gamma \rangle = \langle -p_\gamma | P_{\gamma^{-1}} \rangle$ and hence p_γ is orthogonal to $P_\gamma + P_{\gamma^{-1}}$. In the same way we obtain that if $A_\gamma(w) = w$ then w is orthogonal to $P_\gamma - P_{\gamma^{-1}}$. If A_γ is diagonalizable then either $A_\gamma = -Id$ or it is the reflection in the plane $W := \langle P | p_\gamma \rangle = 0$. In the first case it follows that $P_{\gamma^{-1}} = -P_\gamma$ and hence Σ_γ and $\Sigma_{\gamma^{-1}}$ are disjoint. Consequently γ is hyperbolic or loxodromic (see [5]). In the second case it follows that $P_\gamma + P_{\gamma^{-1}} \in W$, $P_\gamma - P_{\gamma^{-1}}$ is orthogonal to W and γ is elliptic or parabolic if and only if $W \cap \Sigma_{\Psi(\gamma)} \neq \emptyset$.

Given $\gamma \in \text{PSL}(2, \mathbb{C})$, choose $\gamma_0 \in \text{SU}(2, \mathbb{C})$ such that $\gamma_1 = \gamma_0^{-1} \gamma \gamma_0$ fixes ∞ when acting on \mathbb{H}^3 . Then $c(\gamma_1) = 0$, $\Sigma_{\gamma_1} = \Sigma_{\gamma \gamma_0} = \gamma_0^{-1}(\Sigma_\gamma)$ and $(\Sigma_\gamma, \Sigma_{\gamma^{-1}}) = (\Sigma_{\gamma \gamma_0}, \Sigma_{\gamma^{-1} \gamma_0}) = \frac{|1 - \langle P_{\gamma \gamma_0} | P_{\gamma^{-1} \gamma_0} \rangle|}{R_{\gamma \gamma_0}^2} = \frac{1}{2(\|\gamma\|^2 - 2)}[-2|b|^2(\text{Re}(a\bar{d}) - 1) - (|a|^2 + |d|^2)(\|\gamma\| - 2)]$.

In case γ is hyperbolic then $a\bar{d} \in \mathbb{R}$ and so $(\Sigma_\gamma, \Sigma_{\gamma^{-1}}) = \frac{a^2 + d^2}{2} = \frac{1}{2}\text{tr}(\gamma^2) = -1 + \frac{1}{2}[\text{tr}(\gamma)]^2$. In case γ is elliptic or parabolic then $d = \bar{a}$ and $|a| = |d| = 1$. Using this we get once more that $(\Sigma_\gamma, \Sigma_{\gamma^{-1}}) = -1 + \frac{1}{2}[\text{tr}(\gamma)]^2$.

Lemma 4.1 *Let $\gamma \in \text{PSL}(2, \mathbb{C})$, be non unitary and non-loxodromic. Then we have that*

1. $(\Sigma_\gamma, \Sigma_{\gamma^{-1}}) = -1 + \frac{1}{2}[\text{tr}(\gamma)]^2$
2. $\sigma_{\gamma^{-1}} \circ \sigma_\gamma = \gamma^2$
3. $A_\gamma^2 = Id$. In particular, A_γ is diagonalizable and it is either $-Id$ or a reflection.

Proof. The first item was proved above. To prove the second item we work in \mathbb{H}^3 and suppose that γ fixes ∞ and $b(\gamma) = 0$ if γ is hyperbolic. We will use the explicit formulas of Σ_γ and σ_γ freely and notation will follow that of the results of Section 2. In the parabolic case we have that $v = b$, $\sigma_\gamma(P) = P - [\langle P | v \rangle + \frac{|v|^2}{2}] \cdot \frac{v}{|v|^2}$ and $\sigma_{\gamma^{-1}}(P) = P - [\langle P | v \rangle - \frac{|v|^2}{2}] \cdot \frac{v}{|v|^2}$. From this it follows that $\sigma_{\gamma^{-1}} \circ \sigma_\gamma(P) = P + 2v = \gamma^2(P)$. In the elliptic case, set $a = a(\gamma) = e^{i\theta}$, $b = b(\gamma) \neq 0$. Then Σ_γ and $\Sigma_{\gamma^{-1}}$ are vertical planes with normal vectors $v_\gamma = e^{-i\theta}b$ and $v_{\gamma^{-1}} = -e^{i\theta}b$, respectively. Hence the angle between these two planes is 2θ , the angle of rotation of γ around its axis, the intersection of the two planes. From this we get that $\sigma_{\gamma^{-1}} \circ \sigma_\gamma = \gamma^2$. In the hyperbolic case we have that $\gamma(P) = a(\gamma)^2 P$, $\sigma_\gamma(P) = \frac{1}{a(\gamma)^2 P}$ and $\sigma_{\gamma^{-1}}(P) = \frac{a(\gamma)^2}{P}$. From this once again we get the desired formula.

To prove the last item recall that $\sigma_{\gamma^{-1}} = A_\gamma \sigma_\gamma A_\gamma$. The second item gives that $\sigma_{\gamma^{-1}} \circ \sigma_\gamma = A_\gamma \sigma_\gamma A_\gamma \sigma_\gamma$ and hence $\sigma_{\gamma^{-1}} = A_\gamma \sigma_\gamma A_\gamma$. It follows that $A_\gamma = A_\gamma^{-1}$, finishing the proof. ■

Lemma 4.2 *Let Σ_1 and Σ_2 be distinct totally geodesic hyper surfaces of \mathbb{B}^3 , $\sigma_i, i = 1, 2$ the reflection in Σ_i and $\gamma = \sigma_1 \circ \sigma_2$. Then γ is non-loxodromic and $(\Sigma_1, \Sigma_2) = \frac{1}{2}|\text{tr}(\gamma)|$.*

Proof. We distinguish three cases: The surfaces are parallel, their intersection is a geodesic or they are disjoint. Working in \mathbb{H}^3 , we may suppose that Σ_2 is the plane $x = x_0$ with $x_0 > 0$.

In the first case Σ_1 is a plane with equation $x = x_1$ with $x_1 \neq x_0$. In this case $\sigma_1 \circ \sigma_2$ is a parabolic element and the formula is easily seen to hold. In fact, the proof is along the same lines as that in the proof of the previous lemma.

In the second case Σ_1 is a vertical plane making an angle of θ degrees with Σ_2 . In this case $\sigma_1 \circ \sigma_2$ corresponds to the rotation of 2θ degrees around $\Sigma_1 \cap \Sigma_2$ and once again the formula holds trivially. Once again we refer to the proof of the previous lemma.

In the third case we may take $\Sigma_1 = S_r(0)$ and $r < x_0$. The action on $\partial\mathbb{H}^3$ is given by $\sigma_1(x) = \frac{r^2}{x}$ and $\sigma_2(x) = -x + 2x_0$. Since $0 < r < x_0$ we find that γ has exactly two fixed points in $\partial\mathbb{H}^3$, x_2 and x_3 say, and that the line segment $[x_2, x_3]$ is invariant under γ . From this it follows that the geodesic line l , in \mathbb{H}^3 , linking x_2 and x_3 is the axis of γ . Mapping l to a vertical line we may suppose that $c(\gamma) = b(\gamma) = 0$. Hence we have that $a(\gamma) \cdot d(\gamma) = 1$, $\gamma(\infty) = \infty$ and $\gamma(0) = 0$. Note that now the Σ_i 's are Euclidean spheres $S_{R_i}(P_i)$. We have that $\infty = \gamma(\infty) = \sigma_1(\sigma_2(\infty))$ and hence $P_2 = \sigma_2(\infty) = P_1 = P$. It follows that $0 = \gamma(0) = \sigma_1(\sigma_2(0)) = [1 + \frac{R_1^4}{R_2^4}]P$ and hence $P = 0$. Consequently $\gamma(X) = \frac{R_1^2}{R_2^2}X$ and thus $a(\gamma)^2 = \gamma(1) = \frac{R_1^2}{R_2^2}$. From this, and the definition of (Σ_1, Σ_2) , the formula follows readily. ■

Lemma 4.3 *Let γ and γ_1 be such that $\Sigma_\gamma = \text{ISO}_\gamma$ and $\Sigma_{\gamma_1} = \text{ISO}_{\gamma_1}$. Then $\{\overline{c(\gamma)}, \overline{ic(\gamma)}, j\}$ is a basis of eigenvectors of A_γ , A_γ is the reflection in the plane $W_\gamma = \text{span}_{\mathbb{R}}[\overline{ic(\gamma)}, j]$ and $\overline{c(\gamma)}$ is orthogonal to W_γ . Moreover, the angle between W_γ and W_{γ_1} is given by $\arg(\frac{c_1}{c_2})$.*

Proof. Suppose that $\Sigma_\gamma = \text{ISO}_\gamma$. Then we have that $P_\gamma = \hat{P}_\gamma$ and $A_\gamma(0) = 0$. It follows that A_γ is a linear isometry. We also have that $A_\gamma(P_\gamma) = \gamma(\infty) = \hat{P}_{\gamma^{-1}}$. Since A_γ is a linear orthogonal map reversing orientation we have that $A_\gamma(iP_\gamma) = -i\hat{P}_{\gamma^{-1}}$. A_γ fixes \mathbb{R}^+j point wise. We have that $P_{\gamma^{-1}} = -\frac{\bar{a}^2}{|a|^2}P_\gamma$. Since $\text{tr}(\gamma) \in \mathbb{R}$ we have that γ is non-loxodromic and hence $A_\gamma^2 = Id$. We have that $A_\gamma(aP_\gamma) = \bar{a}P_{\gamma^{-1}} = -aP_\gamma$ and hence $A_\gamma(iaP_\gamma) = iaP_\gamma$. Since \bar{c} is an \mathbb{R} -multiple of aP_γ , it follows that $A_\gamma(\bar{c}) = -\bar{c}$ and hence $\{\bar{c}, i\bar{c}, j\}$ is a basis of eigenvectors of A_γ . So A_γ is the reflection in the plane $W_\gamma = \text{span}_{\mathbb{R}}[i\bar{c}, j]$ and \bar{c} is orthogonal to W_γ . Hence the angle between W_γ and W_{γ_1} is given by $\arg(\frac{c_1}{c_2})$. ■

It would be interesting to know if $\arg(\frac{c_1}{c_2})$ is also the dihedral angle between Σ_γ and Σ_{γ_1} . This is true in the examples given in [8]. Note also that γ is hyperbolic, elliptic or parabolic if either $\Sigma_\gamma \cap W_\gamma$ is empty, is a circle or consists of a single point. This follows also from the description of the relative position of Σ_γ and $\Sigma_{\gamma^{-1}}$ given in [5]. The lemma also suggests that a DF domain must be quite symmetrical. In the Fuchsian case the symmetry is with respect to the i -axes in \mathbb{H}^2 (see [5, 8]).

In the Fuchsian case we take \mathbb{B}^2 as our model but, for simplicity of notation, use the notation of \mathbb{H}^2 . Let $\gamma = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ with $|a|^2 - |b|^2 = 1$ and $b \neq 0$. Writing $\gamma = A_\gamma \sigma_\gamma$ we have that $A_\gamma(P_\gamma) = P_{\gamma^{-1}}$. Here we have that $P_\gamma = \frac{-\bar{a}}{b}$. Denote by $\{P_2, P_1\}$ and $\{P_4, P_3\}$, respectively the intersections $\Sigma_\gamma \cap S_1(0)$ and $\Sigma_{\gamma^{-1}} \cap S_1(0)$ (reading counterclockwise). The points $P_k, k = 1, 4$ can be obtained solving the equations $a\bar{b}z^2 + 2|b|^2z + \bar{a}b = 0$ and $\bar{a}bz^2 - 2|b|^2z + ab = 0$. We obtain that $P_1 = \lambda P_\gamma$, $P_4 = \bar{\lambda} P_{\gamma^{-1}}$, where $\lambda = \frac{|b|}{|a|^2}(|b| + i)$, and, since A_γ reverses orientation, $A_\gamma(P_1) = P_4$. Actually since A_γ is an orthogonal map reversing orientation and $A_\gamma(P_\gamma) = P_{\gamma^{-1}}$, we have that $A_\gamma(iP_\gamma) = -iP_{\gamma^{-1}}$. Noting that $P_{\gamma^{-1}} = -\frac{a^2}{|a|^2}P_\gamma$, and using the linearity of A_γ , we obtain that $A_\gamma^2 = Id$. In fact, $P_\gamma + P_{\gamma^{-1}}$ or $P_\gamma - P_{\gamma^{-1}}$ is an eigenvector of A_γ . If necessary, the other eigenvector is obtained multiplying the one already obtained by i . Hence A_γ is always diagonalizable and is a reflection. We summarize this in the following result.

Lemma 4.4 *Let $\gamma = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$, with $|a|^2 - |b|^2 = 1$ and $b \neq 0$, act on \mathbb{B}^2 . Then A_γ is a reflection and $P_\gamma + P_{\gamma^{-1}}$ or $P_\gamma - P_{\gamma^{-1}}$ is an eigenvector of A_γ .*

If Σ_1 and Σ_2 are hyperbolic planes orthogonal to one another then the product of the reflections in Σ_1 and Σ_2 is the reflection in their intersection.

Lemma 4.5 *Let L_1 and L_2 be two hyperbolic lines, σ_k the reflection in $L_k, k = 1, 2$ and $\gamma = \sigma_1 \circ \sigma_2$. Then the following hold.*

1. $\text{tr}(\gamma) \in \mathbb{R}$ if and only if there exist a hyperbolic plane Σ containing both L_1 and L_2 .
2. Every hyperbolic or elliptic transformation is a product of two reflections in a line.
3. If $L_1 \cup L_2 \subset \Sigma$, where Σ is a hyperbolic plane, then γ is parabolic, elliptic or hyperbolic depending on the two lines being tangent, intersecting or disjoint. If such a Σ does not exist then γ is loxodromic.

Proof. We may suppose that L_1 is the j -axis and that L_2 is the line joining the points z_0 and z_1 in $\partial\mathbb{H}^3$. In this case we have that $\sigma_1 = \begin{pmatrix} i & o \\ 0 & -i \end{pmatrix}$ and $\sigma_2 = \gamma_1^{-1} \sigma_1 \gamma_1$, where $\gamma_1 = \begin{pmatrix} 1 & -z_0 \\ 1 & -z_1 \end{pmatrix}$. Write $z_1 = z_0 + 2Re^{i\theta}$, where R is the radius of L_2 . Then $\text{tr}(\gamma) = -\frac{z_0 e^{-i\theta} + R}{R}$. It follows that $\text{tr}(\gamma) \in \mathbb{R}$ if and only if $z_0 = te^{i\theta}, t \in \mathbb{R}$ and hence $\frac{z_0}{z_1} \in \mathbb{R}$. Hence both lines are contained in a vertical plane and $|\text{tr}(\gamma)| = 2|1 + \frac{t}{R}|$. From this it follows that γ is parabolic if $t \in \{0, -2R\}$, elliptic if $-2R < t < 0$ and hyperbolic if $t \notin [-2R, 0]$. So from now on we may suppose that $\theta = 0$, i.e., $z_0, z_1 \in \mathbb{R}$.

In the hyperbolic case we may suppose $z_0 = 1$. The set of eigenvalues of γ is $\{\frac{1-\sqrt{z_1}}{1+\sqrt{z_1}}, \frac{1+\sqrt{z_1}}{1-\sqrt{z_1}}\}$ and the fixed points are $\pm\sqrt{z_1}$. The image of the function $f(z_1) = \frac{1+\sqrt{z_1}}{1-\sqrt{z_1}}$ is $]0, 1[$ and hence every hyperbolic element is a product of two reflections. Note that γ restricted to L_2 is an Euclidean isometry and hence $L_2 \subset \Sigma_\gamma$. Note also that in this case the axis of γ , the hyperbolic line linking its fixed points, is orthogonal to L_1 .

Finally, we consider the elliptic case. In this case we have that $z_1 > 0$ and hence we may suppose $t = -1$. We obtain that the fixed points of γ are $\pm i\sqrt{z_1}$ and its spectrum is $\{\lambda_1, \lambda_2\}$ with $\lambda_2 = \frac{2\sqrt{z_1}}{z_1+1} + i\frac{z_1-1}{z_1+1}$. The image of the function $f(z_1) = \frac{z_1-1}{z_1+1}$ is $] -1, 1[$ and hence each elliptic

element is the product of two reflections in a line. Note that the axis of γ (the line linking the two fixed points) and the two lines, L_1 and L_2 , all passes through the point $\sqrt{z_1}j$. ■

One can consider also the composition of the rotations in two lines. The situation is a bit more complicated but can be handled in a similar way.

Given $\gamma \in \text{PSL}(2, \mathbb{C})$ define the *canonical region* of γ to be $\text{Canreg}(\gamma) := \{P \in \mathbb{H}^3 \mid \sinh[\frac{1}{2}\rho(P, \gamma(P))] < \frac{1}{2}|\text{tr}(\gamma)|\}$ if $o(\gamma) \neq 2$ and $\text{Canreg}(\gamma) := \text{Fix}(\gamma)$ if $o(\gamma) = 2$ (see [1] for the Fuchsian case). Then clearly $\text{Canreg}(\gamma_0\gamma\gamma_0^{-1}) = \gamma_0(\text{Canreg}(\gamma))$. From this it follows that $\gamma(\text{Canreg}(\gamma)) = \text{Canreg}(\gamma)$.

In \mathbb{H}^3 we have that if $P = z + rj$ then $\sinh[\frac{1}{2}\rho(P, \gamma(P))] = \frac{\|P - \gamma(P)\|}{2r}$. From this it follows easily that in the parabolic case, with γ stabilizing ∞ , $\text{Canreg}(\gamma)$ is the horoball $\{P = z + rj \in \mathbb{H}^3 \mid r > \frac{|b(\gamma)|}{2}\}$.

In the elliptic case we may suppose that γ is a diagonal matrix and $a = a(\gamma) = e^{i\theta}$. In this case $\sinh[\frac{1}{2}\rho(P, \gamma(P))] = \frac{|1-a^2||z|}{2r} = \frac{|z|}{r}|\sin(\theta)| = \frac{|z|}{r}|\sin(\ln(a))|$. Let $L = \mathbb{R}j$. We have that $\cosh(\rho(P, \|P\|j)) = \frac{\|P\|}{r}$ and hence $\sinh(\rho(P, \|P\|j)) = \frac{\|z\|}{r}$. It follows that $\sinh[\frac{1}{2}\rho(P, \gamma(P))] = \sinh(\rho(P, L))|\sin(\ln a)|$.

We now look at the hyperbolic case. Proceeding as in the elliptic case we obtain that $\sinh[\frac{1}{2}\rho(P, \gamma(P))] = \frac{\|P\|}{r}|\sinh(\ln a)|$. In a similar way we obtain that $\sinh[\frac{1}{2}\rho(P, \gamma(P))] = \sinh(\rho(P, L))|\sinh(\ln a)|$. In this case $\frac{1}{2}|\text{tr}(\gamma)| = \cosh(\ln a)$ and hence $P = z + rj = x + yi + rj \in \text{Canreg}(\gamma)$ if $x^2 + y^2 < \frac{r^2}{\sinh^2(\ln a)}$. Consider the line l_γ given by the equations $y = 0$ and $r - x \sinh(\ln a) = 0$ and also the sphere, Σ say, passing through z and $\gamma(z)$ and orthogonal to $\partial(\mathbb{H}^3)$. Then Σ is the sphere with center $\frac{(1+a^2)z}{2}$ and radius $\frac{|1-a^2||z|}{2}$. A simple calculation shows that Σ is tangent to $\text{Canreg}(\gamma)$. In this case, note that if $\text{Canreg}(\gamma) = \text{Canreg}(\gamma_1)$ then $|\sinh(\ln a(\gamma))| = |\sinh(\ln a(\gamma_1))|$. From this it readily follows that $\gamma_1 \in \{\gamma, \gamma^{-1}\}$.

In the elliptic case we obtain the cone $x^2 + y^2 < r^2 \coth^2(\theta_\gamma)$, where $a(\gamma) = e^{i\theta}$. From this we also infer that if $\text{Canreg}(\gamma) = \text{Canreg}(\gamma_1)$ then $|\tan(\theta_\gamma)| = |\tan(\theta_{\gamma_1})|$ and hence $\gamma_1 \in \{\gamma, \gamma^{-1}\}$. In both cases L is the axis of γ .

Acknowledgment: The first author is grateful to the Universidade Federal da Paraíba (UFPB-Brazil) and the Universidade Federal do Vale do São Francisco (UNIVASF-Brazil), for their hospitality while this research was being done.

References

- [1] A. F. Beardon, The Geometry of Discrete Groups, Springer Verlag NY, 1983.
- [2] M. R. Bridson, A. Haefliger, Metric Spaces of Non-Positive Curvature, Springer Verlag, Berlin, 1999.
- [3] J. Elstrodt, F. Grunewald, J. Mennicke, Groups Acting on Hyperbolic Space, Springer Verlag, Berlin Heidelberg, 1998.
- [4] M. Gromov, Hyperbolic Groups, in Essays in Group Theory, M. S. R. I. Publ. 8, Springer Verlag, 1987, 75-263.
- [5] E. Jespers, S.O. Juriaans, A. Kiefer, A. De A. E Silva, A.C. Souza Filho, Poincaré Bisectors in Hyperbolic Spaces, Submitted.

- [6] S. Johansson, On fundamental domains of arithmetic Fuchsian Groups, Math. Comp. 69 (2000),no.229, 339-349.
- [7] S. Katok, Reduction Theory for Fuchsian Groups, Math. Ann. 273, 461-470 (1986).
- [8] G. S. Lakeland, Dirichlet-Ford Domains and Arithmetic Reflection Groups, Pacific J. Math., to appear.
- [9] J.G., Ratcliffe, Foundations of Hyperbolic Manifolds, Springer Verlag, New York, 1994.
- [10] R. Swan, Generators and relations for certain special linear groups, Adv. in Math., v. 6, 1971, pp. 1-77.

Instituto de Matemática e Estatística,
 Universidade de São Paulo (IME-USP),
 Caixa Postal 66281, São Paulo,
 CEP 05315-970 - Brasil
 email: ostanley@usp.br

Universidade Federal do Vale do São Francisco,
 Colegiado de Engenharia Mecânica,
 Avenida Antonio Carlos Magalhães, 510,
 Colegiado de Engenharia Mecânica,
 Santo Antônio
 48902-300 - Juazeiro, BA - Brasil.
 e-mail: cirino.lima@univasf.edu.br

Departamento de Matemática
 Universidade Federal da Paraíba
 João Pessoa - PB - Brasil
 e-mail: andrade@mat.ufpb.br